

A CHARACTERIZATION OF CAUSAL AUTOMORPHISMS ON TWO-DIMENSIONAL MINKOWSKI SPACETIME

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ABSTRACT. It is shown that causal automorphisms on two-dimensional Minkowski spacetime can be characterized by the invariance of the wave equations.

1. INTRODUCTION

In 1964, Zeeman clarified general forms of causal automorphisms on \mathbb{R}_1^n with $n \geq 3$. (Ref. [1]) As the title of his paper says, the result is that causality implies the Lorentz group.

However, this is not the case for \mathbb{R}_1^2 . In 2010, general forms of causal automorphisms on \mathbb{R}_1^2 is given and the result gives us much more symmetry than Zeeman's result. (Ref. [2], [3], [4] and [5])

From this result, we can see that each component of the causal automorphisms satisfies the wave equation and so a new question is raised by Low. (Ref. [4]) His question is essentially how to characterize causal automorphisms by wave equations. A characterization of causal automorphisms on \mathbb{R}_1^n with $n \geq 3$ is given in Ref. [6]. The result is that causal automorphisms on \mathbb{R}_1^n with $n \geq 3$ can be characterized by the invariance of the wave equations. (Ref. [6]). Physically, the invariance of the wave equations means the principle of the constancy of the speed of the light, which is the second postulate of special relativity. Therefore, we can say that Einstein's second postulate implies the preservation of causal relations as well as the Lorentz group, when the dimensions of spacetimes are greater than or equal to 3.

In this paper, it is shown that C^2 -causal automorphisms on \mathbb{R}_1^2 can be obtained by the invariance of the wave equations. Therefore, in conclusion, we can say that causal automorphisms on Minkowski spacetime can be characterized as the invariance of the wave equations, regardless of the spacetime dimension.

Key words and phrases. causal isomorphism, wave equation, causal relation.

2. PRELIMINARIES

In Ref. [3], the following is shown.

Theorem 2.1. *Let $F : \mathbb{R}_1^2 \rightarrow \mathbb{R}_1^2$ be a causal automorphism. Then, there exists unique homeomorphisms φ and ψ of \mathbb{R} , which are both increasing or both decreasing such that if φ and ψ are increasing, then we have $F(x, t) = \frac{1}{2}(\varphi(x+t) + \psi(x-t), \varphi(x+t) - \psi(x-t))$, of if φ and ψ are decreasing, then we have $F(x, t) = \frac{1}{2}(\varphi(x-t) + \psi(x+t), \varphi(x-t) - \psi(x+t))$.*

Conversely, for any given homeomorphisms φ and ψ of \mathbb{R} , which are both increasing or both decreasing, the function F defined as above is a causal automorphism on \mathbb{R}_1^2 .

Proof. See Theorem 2.2 in Ref. [3]. \square

In Ref. [4], Low has shown that if we use null coordinates, the above result can be simplified.

If we use null coordinate system as $u = x + t$, $v = x - t$, then the above Theorem can be expressed as following.

Theorem 2.2. *Let $(U, V) = F(u, v)$ be a causal automorphism on \mathbb{R}_1^2 . Then there exist unique homeomorphisms φ and ψ on \mathbb{R} , which are both increasing, or both decreasing such that, if φ and ψ are increasing, then we have $F(u, v) = (\varphi(u), \psi(v))$, or if φ and ψ are decreasing, then we have $F(u, v) = (\varphi(v), \psi(u))$.*

Conversely, for any given homeomorphisms φ and ψ of \mathbb{R} , which are both increasing or both decreasing, the function F defined as above is a causal automorphism on \mathbb{R}_1^2 .

3. INVARIANCE OF WAVE EQUATIONS

If we use null coordinates u and v , then the wave equation $\frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial t^2} = 0$ can be simplified by $\frac{\partial^2 \theta}{\partial u \partial v} = 0$.

We now prove the main Theorem.

Theorem 3.1. *Let $F : \mathbb{R}_1^2 \rightarrow \mathbb{R}_1^2$ be a C^2 diffeomorphism given by $(\sigma, \tau) = F(u, v)$ in terms of null coordinates. Then the necessary and sufficient condition for F to be a causal automorphism is that $\frac{\partial(\sigma-\tau)}{\partial t} \geq 0$, $\sigma_u \tau_v + \sigma_v \tau_u \geq 0$ and, for any C^2 function θ on \mathbb{R}_1^2 , $\theta_{uv} = 0$ if and only if $\theta_{\sigma\tau} = 0$.*

Proof. For the proof of the necessary condition, we only need straightforward calculations and Theorems in Section 2 and so we omit the proof.

To prove sufficient part we note that, since F is a diffeomorphism, we can use σ and τ as a new coordinate system on \mathbb{R}_1^2 and its Jacobian $J(F) = \sigma_u \tau_v - \sigma_v \tau_u$ is nowhere zero.

Step 1 : By the chain rule, we have the following relation.

$$\begin{aligned} \frac{\partial^2 \theta}{\partial v \partial u} &= \left(\frac{\partial^2 \theta}{\partial \sigma^2} \frac{\partial \sigma}{\partial v} \frac{\partial \sigma}{\partial u} + \frac{\partial^2 \theta}{\partial \tau \partial \sigma} \frac{\partial \tau}{\partial v} \frac{\partial \sigma}{\partial u} + \frac{\partial \theta}{\partial \sigma} \frac{\partial^2 \sigma}{\partial v \partial u} \right) \\ &+ \left(\frac{\partial^2 \theta}{\partial \sigma \partial \tau} \frac{\partial \sigma}{\partial v} \frac{\partial \tau}{\partial u} + \frac{\partial^2 \theta}{\partial \tau^2} \frac{\partial \tau}{\partial v} \frac{\partial \tau}{\partial u} + \frac{\partial \theta}{\partial \tau} \frac{\partial^2 \tau}{\partial v \partial u} \right) \dots\dots (*) \end{aligned}$$

If we let $\theta = \sigma$, then it satisfies $\theta_{\sigma\tau} = 0$ and thus it must satisfy $\theta_{uv} = 0$. If we put this into the above equation, we have $\frac{\partial^2 \sigma}{\partial v \partial u} = 0$. Since the coordinate transformation F is C^2 , we have $\sigma = \int f(u)du + g(v)$.

Likewise, if we put $\theta = \tau$ in the above equation, we obtain $\frac{\partial^2 \tau}{\partial v \partial u} = 0$ and so we have $\tau = \int h(u)du + j(v)$.

If we put $\frac{\partial^2 \sigma}{\partial v \partial u} = 0$ and $\frac{\partial^2 \tau}{\partial v \partial u} = 0$ into $(*)$, we have the following.

$$\frac{\partial^2 \theta}{\partial v \partial u} = \left(\frac{\partial^2 \theta}{\partial \sigma^2} \frac{\partial \sigma}{\partial v} \frac{\partial \sigma}{\partial u} + \frac{\partial^2 \theta}{\partial \tau \partial \sigma} \frac{\partial \tau}{\partial v} \frac{\partial \sigma}{\partial u} \right) + \left(\frac{\partial^2 \theta}{\partial \sigma \partial \tau} \frac{\partial \sigma}{\partial v} \frac{\partial \tau}{\partial u} + \frac{\partial^2 \theta}{\partial \tau^2} \frac{\partial \tau}{\partial v} \frac{\partial \tau}{\partial u} \right) \dots\dots (**)$$

Step 2 : If we let $\theta = \sigma^2$, then it satisfies $\theta_{\sigma\tau} = 0$ and thus it must satisfy $\theta_{uv} = 0$. If we put this into $(**)$, we have $\frac{\partial \sigma}{\partial v} \frac{\partial \sigma}{\partial u} = 0$. Since $\sigma = \int f(u)du + g(v)$, we have $f(u) \cdot g'(v) = 0$.

Likewise, if we put $\theta = \tau^2$ into $(**)$, we have $\frac{\partial \tau}{\partial v} \frac{\partial \tau}{\partial u} = 0$. Since $\tau = \int h(u)du + j(v)$, we have $h(u) \cdot j'(v) = 0$.

Step 3 : We consider two separate cases.

Case (i) : We assume that there exists u_0 such that $f(u_0) \neq 0$. Then, since $f(u) \cdot g'(v) = 0$ for all u and v , we must have $g'(v) = 0$ for all v . Then, since the Jacobian $J(F) = f(u)j'(v) - h(u)g'(v)$ is nowhere zero, $f(u)$ must be nowhere zero. Also, by the same reason, $j'(v)$ is nowhere zero and thus $j(v)$ is a homeomorphism. Since $h(u) \cdot j'(v) = 0$, $h(u) = 0$ for all u . In conclusion, we have $\sigma = \int f(u)du$ and $\tau = j(v)$, which are homeomorphisms.

Case (ii) : We now assume that f is identically zero. Then, since $J(F)$ is nowhere zero, $g'(v)$ is nowhere zero and thus $g(v)$ is a homeomorphism. Since $J(F)$ is nowhere zero, $h(u)$ must be nowhere zero and thus from $h(u)j'(v) = 0$, we have $j'(v) = 0$ for all v . In conclusion, we have $\sigma = g(v)$ and $\tau = \int h(u)du$, which are homeomorphisms.

Step 4 : We now consider the condition $\frac{\partial(\sigma-\tau)}{\partial t} \geq 0$ which implies time-orientation preservation. By chain rule, we have the following.

$$\begin{aligned}
0 \leq \frac{\partial(\sigma - \tau)}{\partial t} &= \frac{\partial\sigma}{\partial t} - \frac{\partial\tau}{\partial t} \\
&= \left(\frac{\partial\sigma}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial\sigma}{\partial v} \frac{\partial v}{\partial t} \right) - \left(\frac{\partial\tau}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial\tau}{\partial v} \frac{\partial v}{\partial t} \right) \\
&= \left(\frac{\partial\sigma}{\partial u} - \frac{\partial\sigma}{\partial v} \right) - \left(\frac{\partial\tau}{\partial u} - \frac{\partial\tau}{\partial v} \right) \\
&= \left(f(u) - g'(v) \right) - \left(h(u) - j'(v) \right)
\end{aligned}$$

Therefore, we have $f(u) + j'(v) \geq h(u) + g'(v)$.

Step 5: We consider two separate cases.

Case (i) : We assume that there exists u_0 such that $f(u_0) \neq 0$. Then, from Step 3, we know that g' and $h(u)$ are identically zero and so from Step 4, we have $f(u) + j'(v) \geq 0$. Since $\sigma_u \tau_v + \sigma_v \tau_u \geq 0$, we have $f(u) \cdot j'(v) \geq 0$. Therefore, both f and j' are positive functions and so $\sigma = \int f(u) du$ and $\tau = j(v)$ are increasing homeomorphisms.

Case (ii) : We now assume that f is identically zero. Then, from Step 3, we know that f and j' are identically zero and so from step 4, we have $h(u) + g'(v) \leq 0$. Since $\sigma_u \tau_v + \sigma_v \tau_u \geq 0$, we have $h(u) \cdot g'(v) \geq 0$. Therefore, both h and g' are negative functions and so $\sigma = g(v)$ and $\tau = \int h(u) du$ are decreasing homeomorphisms.

Step 6 : From Theorem 2.2, $(\sigma, \tau) = F(u, v)$ is a causal automorphism on \mathbb{R}_1^2 .

□

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